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Oscillator eigenstates concentrated on classical trajectories

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Abstract. The highly degenerate eigenspaces of the two-dimensional isotropic harmonic oscillator are proven to contain eigenstates that are optimally localized on the closed trajectories of the classical dynamics. As $\hbar \to 0$, their phase space probability density converges to the unique probability density on the corresponding trajectory which is invariant under the classical flow.

Résumé. Nous démontrons que les espaces propres très dégénérés de l'oscillateur harmonique isotrope en deux dimensions contiennent des états propres localisés optimalement sur les trajectoires fermées de la dynamique classique. Quand $\hbar \to 0$, leur densité de probabilité dans l'espace des phases converge vers l'unique densité de probabilité sur la trajectoire qui est invariante sous le flot classique.

1. Introduction

The classical orbits of the two-dimensional isotropic hamonic oscillator are ellipses with the origin at their centre. At fixed energy they form a two-parameter family. The energy spectrum of the corresponding quantum oscillator is given by

$$E_N = \hbar\omega(N+1) \tag{1.1}$$

with each energy eigenvalue $(N+1)$-fold degenerate. We will show in this paper how to associate with each classical orbit at energy E a particular energy eigenstate of the quantum oscillator with the same energy in such a way that, as $\hbar \to 0$, the eigenstate completely concentrates on the corresponding classical trajectory (theorem 4.1). We also explicitly compute the first-order term in the asymptotic expansion in $\hbar$.

The classical limit of energy eigenstates has been studied extensively in the literature, in particular for completely integrable systems [1, 11]. Let $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^{n*}$ be the classical phase space and $\boldsymbol{P} : T^*\mathbb{R}^n \to \mathbb{R}^n$ n commuting constants of the motion for the Hamiltonian H, i.e.

$$\{P_i, P_j\} = 0 \tag{1.2a}$$

and

$$H = H(\boldsymbol{P}). \tag{1.2b}$$

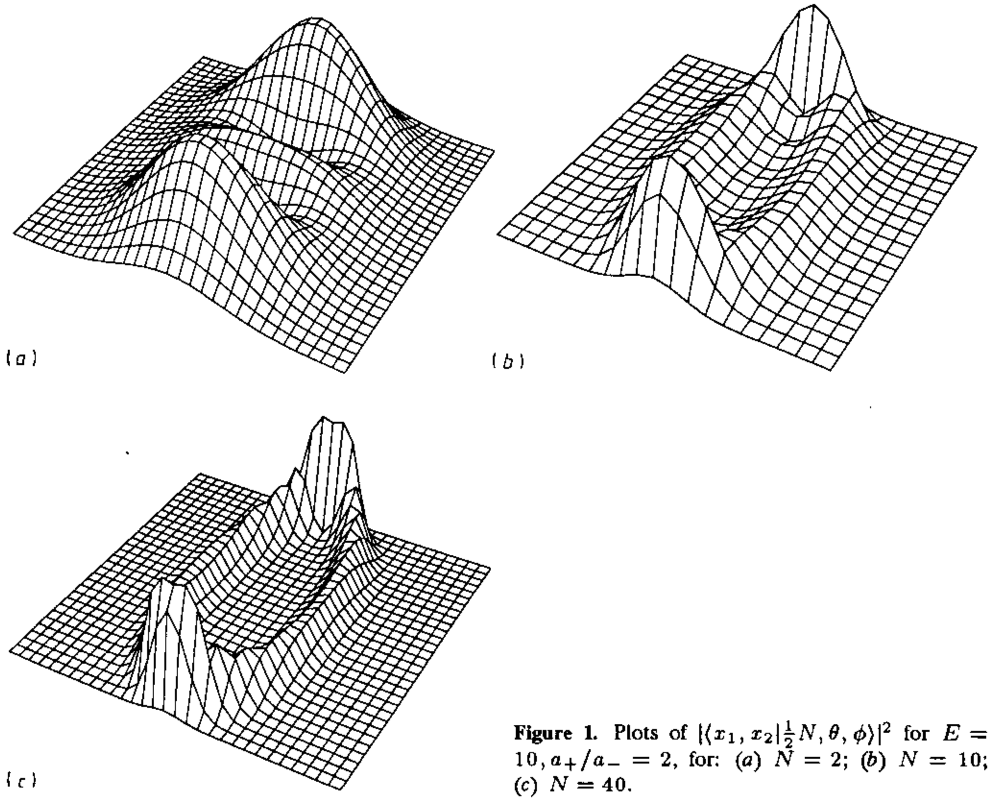

Figure 1. Plots of $|\langle x_1, x_2|\frac{1}{2}N, \theta, \phi\rangle|^2$ for $E = 10, a_+/a_- = 2$, for: (*a*) $N = 2$; (*b*) $N = 10$; (*c*) $N = 40$.

In the corresponding quantum system, one expects the quantized P_i's to form a complete set of commuting observables on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. As a result, fixing their eigenvalues λ_i determines a unique eigenstate of the quantum Hamiltonian H and one expects that, as $\hbar \to 0$, this eigenstate concentrates—in phase space—uniformly on the corresponding classical torus $\boldsymbol{P}^{-1}(\boldsymbol{\lambda})$. This is indeed established in [1], under suitable conditions on H. It is in particular assumed in [1] that one stays away from critical values of $\boldsymbol{P}$ (i.e. points where the rank of the $2n \times n$ matrix of partial derivatives of $\boldsymbol{P}$ is strictly less than n).

We show that the two-dimensional isotropic harmonic oscillator has certain eigenstates that do not conform to this general pattern, since they localize on one-dimensional trajectories, rather than on two-dimensional tori. More precisely, our result is the following. At fixed E, the classical orbits are indexed by two parameters, fixing the length of the two major axes of the ellipse, and the angle between the major axis and the x_1-axis. As explained in section 2, we can choose the above parameters to be $\theta \in [0, 2\pi)$ and $\phi \in [0, 2\pi)$ (see (2.9)–(2.10)). Setting $E = E_N$, we construct in section 3 for each classical orbit characterized by E, θ, ϕ a family of eigenstates $|\frac{1}{2}N, \theta, \phi\rangle$ indexed by N (see (3.13)–(3.15)). In the limit $\hbar \to 0, N \to \infty, E_N = E$, those satisfy (theorem 4.1)

$$\langle \tfrac{1}{2}N; \theta, \phi | F_\hbar(X, P) | \tfrac{1}{2}N; \theta, \phi\rangle \to \frac{1}{T}\int_0^T \mathrm{d}\tau\, f(x(\tau), p(\tau)) \tag{1.3}$$

where $t \in [0, T] \to (x(t), p(t)) \in T^*\mathbb{R}^n$ is the trajectory considered and $T = 2\pi/\omega$ its period (see (2.7)). The expression in the right-hand side of (1.3) is the time average of the classical observable f on the trajectory. On the left $F_\hbar(X, P)$ is the anti-Wick ordered quantization of f (see [9] and (4.13)). Note that $\mathrm{d}t/T$ is the unique flow-invariant probability measure on the trajectory. The result in (1.3) implies that the x-space probability density $|\langle x_1, x_2|\frac{1}{2}N, \theta, \phi\rangle|^2$ is increasingly concentrated on the classical ellipse $t \rightarrow x(t)$, as $N \rightarrow \infty$, and with an intensity that is inversely proportional to the speed $\|\mathrm{d}x/\mathrm{d}t\|$. This effect is clearly exhibited in figure 1; in particular the high concentration far from the ellipse's centre can be noticed, where the motion is slow. Similar pictures could be obtained for the momentum space density.

To understand why this does not contradict the results in [1], it suffices to remark that, although the eigenstates we consider are indeed simultaneous eigenstates of two commuting constants of the motion P_1 and P_2, the corresponding eigenvalues constitute a critical value of $\boldsymbol{P}$. The eigenfunction still concentrates on the corresponding level set of $\boldsymbol{P}$, which is, however, one-dimensional rather than two-dimensional as in the regular case. We note that, in general, one can only expect to be able to associate approximate eigenfunctions (quasi-modes) to closed classical trajectories (see for example [10] and references therein).

From a different point of view, the results in this paper are related to the exceptional nature of the system, which is maximally super-integrable ([7] and references therein): this means it admits the maximum number possible, i.e. three, functionally independent constants of the motion J_1, J_2, J_3 (see (2.3)). The SU(2) algebra they generate yields the hidden symmetry group of the system and it is at the origin of the degeneracies in (1.1). Each eigenspace $\mathcal{H}_N$ of energy E_N carries an irreducible representation of 'spin' $N/2$. The particular eigenstates in $\mathcal{H}_N$ considered in this paper turn out to be SU(2) coherent states, obtained by applying the SU(2) representation to a minimal weight vector in $\mathcal{H}_N$ [9].

The construction here is the analogue of the one used in [4, 8] to construct eigenstates in the Kepler problem, optimally localized on classical Kepler ellipses. Results analogous to theorem 4.1 are obtained numerically in [4, 8]. One could also envisage studying in a similar way the eigenstates associated with non-maximally super-integrable systems [7], where certain eigenstates should localize on lower-dimensional subsets of classical n-dimensional tori.

We note also that an oscillator in two dimensions with two different frequencies ω_1 and ω_2 which are independent over the rationals has only two periodic orbits at each energy, when $x_1 = 0$, $p_1 = 0$, or when $x_2 = 0$, $p_2 = 0$. It is not difficult to see that the corresponding eigenstates that will localize on those trajectories are the ones where all energy is in one degree of freedom, the other one being in its ground state. The case when the two frequencies are commensurate but not equal is more complicated, and we will return to it in a later publication [3].

The paper is organized as follows. In section 2 we describe the spaces of classical trajectories of the classical oscillator, and in section 3 we construct a stationary state of the quantum Hamiltonian associated with each such trajectory. In section 4 we study the semiclassical behaviour of those states.

2. The classical trajectory space

Let

$$H(x,p) = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}m\omega^2(x_1^2 + x_2^2) \tag{2.1}$$

be the Hamiltonian of an isotropic two-dimensional harmonic oscillator. We write $(x,p) = (x_1, p_1, x_2, p_2) \in \mathbb{R}^4$ for points in phase space and we introduce the energy surface

$$\Sigma_E = \{(x,p) \in \mathbb{R}^4 | H(x,p) = E\}. \tag{2.2}$$

The previous Hamiltonian has three independent constants of the motion:

$$J_1 = \frac{1}{2\omega}(H_1 - H_2) \tag{2.3a}$$

$$J_2 = \frac{1}{2\omega}\left(\frac{1}{m}p_1 p_2 + m\omega^2 x_1 x_2\right) \tag{2.3b}$$

$$J_3 = \tfrac{1}{2}(x_1 p_2 - x_2 p_1) = \tfrac{1}{2}L \tag{2.3c}$$

where we wrote $H_i = [(1/2m)p_i^2 + \frac{1}{2}m\omega^2 x_i^2]$ for the energy of the ith component. They satisfy the commutation relations of the SU(2) algebra

$$\{J_i, J_j\} = \epsilon_{ijk} J_k \tag{2.4}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket on $\mathbb{R}^4$. Furthermore,

$$J_1^2 + J_2^2 + J_3^2 = \tfrac{1}{4}H^2\omega^{-2}. \tag{2.5}$$

Now consider the function

$$J : (x,p) \in R^4 \longrightarrow J(x,p) \in \mathbb{R}^3. \tag{2.6}$$

In view of (2.5), the image of Σ_E under J is a sphere of radius $H/2\omega$ in $\mathbb{R}^3$. Since J is constant along any flow line of the Hamiltonian H, each such flow line is mapped by J into a point of this sphere. Conversely, as we shall see, to each point on this sphere corresponds exactly one flow line of H in the phase space $\mathbb{R}^4$. To see this, recall first that the general solution of the equations of motion is

$$x_1(t) = x_1 \cos\omega t + \frac{p_1}{m}\omega^{-1}\sin\omega t \tag{2.7a}$$

$$x_2(t) = x_2 \cos\omega t + \frac{p_2}{m}\omega^{-1}\sin\omega t \tag{2.7b}$$

$$p_1(t) = p_1 \cos\omega t - m\omega x_1 \sin\omega t \tag{2.7c}$$

$$p_1(t) = p_2 \cos\omega t - m\omega x_2 \sin\omega t. \tag{2.7d}$$

From (2.7*a*, *b*), a calculation gives, for $J_3 \neq 0$,

$$(x_1(t)x_2(t))\begin{pmatrix} H_2\omega^{-1} & -J_2 \\ -J_2 & H_1\omega^{-1}\end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix} = (m\omega)^{-1}\frac{L^2}{2}. \tag{2.8}$$

From (2.8) one concludes that the trajectory in the (x_1, x_2) plane is an ellipse. The lengths of its two major axes are given by

$$a_\pm = \left[\frac{L^2}{(m\omega)[H\omega^{-1} \mp ((H\omega^{-1})^2 - L^2)^{1/2}]}\right]^{1/2}. \tag{2.9}$$

Note that the orbit is circular iff $L = \pm H\omega^{-1}$, which implies, in view of (2.5), that $J_2 = 0 = J_1$. From (2.8) we also see that the axes of the ellipse are lined up with the coordinate axes iff $J_2 = 0$. More generally, to determine the position of the ellipse axes, we proceed as follows. Let

$$J_3 = (H/2\omega)\cos\theta \tag{2.10a}$$

$$J_2 = (H/2\omega)\sin\theta\sin\phi \tag{2.10b}$$

$$J_1 = (H/2\omega)\sin\theta\cos\phi. \tag{2.10c}$$

Then

$$\Phi = (\phi/2) \in [0, \pi) \tag{2.11}$$

is the angle between the major axis of the ellipse and the x_1 axis. To see this, note that for $\phi = 0$, (2.11) is correct. Now, since $J_3 = \frac{1}{2}L$, a rotation of Φ in physical space corresponds to a rotation of ϕ in $\boldsymbol{J}$-space. We conclude that, given a point in phase space, equations (2.9)–(2.11) allow us to completely determine the trajectory of the system in the (x_1, x_2)-plane in terms of the constants of the motion. Note that to each x-space trajectory correspond two phase space trajectories, one for clockwise $(L < 0)$ and one for counterclockwise $(L \rangle 0)$ motion. Finally, for $L = 0$, the ellipse degenerates into a straight line the equation of which is correctly determined from the limiting form of (2.8)–(2.9) as $L \to 0$. Starting from a point (x, p) on a circular trajectory, we can obtain any point on any other trajectory at the same energy by successively applying a transformation generated by J_2 over an angle θ (this will deform the circle into an ellipse), then a rotation in physical space over an angle Φ (to rotate the axes of the ellipse), and finally a time translation using H, to move along the ellipse obtained. The fact that the value of $\boldsymbol{J}$ uniquely determines a phase space trajectory of the system will be used in the next section to construct for each such trajectory a state which is—in some suitable sense—optimally localized on it.

3. Quantum eigenstates associated to classical trajectories

The Hamiltonian for the quantum oscillator can be written as

$$H = \hbar\omega(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) \tag{3.1}$$

with

$$a_i = \frac{1}{\sqrt{2}}\left(\left(\frac{m\omega}{\hbar}\right)^{1/2} x_i + \mathrm{i}(\hbar m\omega)^{-1/2} p_i\right). \tag{3.2}$$

Furthermore, the quantization of (2.3) yields

$$J_1 = \frac{\hbar}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \tag{3.3a}$$

$$J_2 = \frac{\hbar}{2}(a_1 a_2^\dagger + a_1^\dagger a_2) \tag{3.3b}$$

$$J_3 = -\frac{\mathrm{i}\hbar}{2}(a_1^\dagger a_2 - a_1 a_2^\dagger) \tag{3.3c}$$

and

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}\hbar J_k. \tag{3.4}$$

Moreover

$$J_1^2 + J_2^2 + J_3^2 = \frac{\hbar^2}{4}\left[\left(\frac{H}{\hbar\omega}\right)^2 - 1\right]. \tag{3.5}$$

The spectrum of H is given by $\{\hbar\omega(N+1) | N \in N\}$ and has degeneracy $d_N = N+1$. We write $\mathcal{H}_N$ for the corresponding space of eigenfunctions. From (3.4)–(3.5) we conclude that the J_i yield an irreducible representation of 'spin' $\frac{1}{2}N$ on the eigenspace $\mathcal{H}_N$. For the unitary group operators we write

$$U(\boldsymbol{\tau}) \equiv \exp(-\mathrm{i}(\boldsymbol{\tau}\cdot \boldsymbol{J}/\hbar)) \qquad |\boldsymbol{\tau}| \leqslant 4\pi.$$

We also remark that the J_i, restricted to $\mathcal{H}_N$, are bounded operators with eigenvalues $\frac{1}{2}N, \frac{1}{2}N-1, \ldots, -\frac{1}{2}N$. We now introduce, in analogy with (2.6),

$$\boldsymbol{J} : \psi \in \mathcal{H}_N \to \frac{1}{\|\psi\|^2}\langle\psi, \boldsymbol{J}\psi\rangle \in \mathbb{R}^3. \tag{3.6}$$

Note that

$$\boldsymbol{J}(U(\boldsymbol{\tau})\psi)\cdot \boldsymbol{J}(U(\boldsymbol{\tau})\psi) = (\boldsymbol{J}(\psi))^2 \tag{3.7}$$

so that the points $\boldsymbol{J}(U(\boldsymbol{\tau})\psi), |\boldsymbol{\tau}| \leqslant 4\pi$ for ψ fixed, lie on the surface of a sphere. Introducing

$$\Delta\psi \equiv (\Delta J_1)^2 + (\Delta J_2)^2 + (\Delta J_3)^2 \tag{3.8}$$

with $(\Delta J_i)^2 = \|\psi\|^{-2}(\langle\psi, J_i^2\psi\rangle - \langle\psi, J_i\psi\rangle^2)$, we have

$$\Delta\psi = \hbar^2 \tfrac{1}{2}N(\tfrac{1}{2}N+1) - \boldsymbol{J}(\psi)\cdot \boldsymbol{J}(\psi). \tag{3.9}$$

Now $\Delta\psi > 0$, since if $\Delta\psi$ were zero, this would mean ψ is a simultaneous eigenvector of all J_i, which is impossible (if $N \neq 0$). To determine the minimum value of $\Delta\psi$, we proceed as follows (see also [9]). Since it follows from (3.7) and (3.9) that $\Delta(U(\boldsymbol{\tau})\psi) = \Delta\psi$, it suffices to minimize ψ among those wavefunctions in $\mathcal{H}_N$ for which

$$\langle\psi, J_1\psi\rangle = 0 = \langle\psi, J_2\psi\rangle, \langle\psi, J_3\psi\rangle > 0. \tag{3.10}$$

The resulting unique wavefunction is the eigenstate of J_3 with maximal eigenvalue $\frac{1}{2}N$, for which we shall use the standard notation $|\frac{1}{2}N, \frac{1}{2}N\rangle$. Explicitly, this state is given by [2]

$$\langle r, \chi | \tfrac{1}{2}N, \tfrac{1}{2}N\rangle = \left(\frac{m\omega}{\hbar\pi N!}\right)^{1/2} \exp(\mathrm{i}N\chi)\left(\left(\frac{m\omega}{\hbar}\right)^{1/2} r\right)^N \exp\left(-\frac{m\omega}{2\hbar}r^2\right) \tag{3.11a}$$

where we introduced polar coordinates in the (x_1, x_2)-plane, i.e.

$$x_1 = r\cos\chi \qquad x_2 = r\sin\chi. \tag{3.11b}$$

The corresponding minimal dispersion is, from (3.9),

$$\Delta_{\min} = \hbar^2 \tfrac{1}{2}N. \tag{3.12}$$

All other states ψ in $\mathcal{H}_N$ with $\Delta_N = \Delta_{\min}$ are then of the form [9]

$$\begin{aligned} |\tfrac{1}{2}N;\theta,\phi\rangle &\equiv \exp\left(-\mathrm{i}\frac{\phi J_3}{\hbar}\right)\exp\left(-\mathrm{i}\frac{\theta J_2}{\hbar}\right)|\tfrac{1}{2}N,\tfrac{1}{2}N\rangle \\ &= \sum_{m'=-\frac{1}{2}N}^{\frac{1}{2}N}\begin{pmatrix} N \\ \frac{1}{2}N+|m'|\end{pmatrix}^{1/2}\cos^{\frac{1}{2}N+m'}\frac{\theta}{2}\sin^{\frac{1}{2}N-m'}\frac{\theta}{2} \\ &\quad\times\exp(-\mathrm{i}m'\phi)|\tfrac{1}{2}N,m'\rangle \end{aligned} \tag{3.13}$$

where the $|\frac{1}{2}N, m'\rangle$ are obtained by successively applying J_- to $|\frac{1}{2}N,\frac{1}{2}N\rangle$, i.e.

$$J_-|\tfrac{1}{2}N,m'\rangle = [\tfrac{1}{2}N(\tfrac{1}{2}N+1) - m'(m'-1)]^{1/2}|\tfrac{1}{2}N,m'-1\rangle. \tag{3.14}$$

This yields

$$\begin{aligned} \langle r,\chi|\tfrac{1}{2}N,m'\rangle &= (-1)^{\frac{1}{2}N-|m'|}C_{N,2|m'|}L_n^{\alpha}\left(\frac{m\omega}{\hbar}r^2\right)r^{2|m'|} \\ &\quad\times\exp\left(-\frac{m\omega}{2\hbar}r^2\right)\exp(2\mathrm{i}m'\chi) \end{aligned} \tag{3.15a}$$

where

$$\alpha = 2|m'| \tag{3.15b}$$

$$n = (\tfrac{1}{2}N - |m'|) \tag{3.15c}$$

$$C_{N,2|m'|} = \left[\frac{1}{\pi}\left(\frac{m\omega}{\hbar}\right)^{2|m'|+1}\frac{n!}{(n+\alpha)!}\right]^{1/2} \tag{3.15d}$$

and the L_n^α are the generalized Laguerre polynomials. To verify that the $|\frac{1}{2}N, m'\rangle$ indeed satisfy (3.14) (i.e. that the phase $(-1)^{\frac{1}{2}N-|m'|}$ is correctly chosen), it suffices to apply J_- and use the functional relations between the L_n^α.

Since the states $|\frac{1}{2}N;\theta,\phi\rangle$ minimize the dispersion of the three observables that completely determine the classical trajectory, we expect them to be eigenstates of H that are optimally localized on the classical trajectory characterized by the point $J(|\frac{1}{2}N,\theta,\phi\rangle)$ on the sphere of radius $(\hbar N/2)$, i.e. of classical energy $\hbar\omega N$. In the next section we prove this claim by studying the phase space localization of those states.

4. The semiclassical limit

To study the phase space localization of the states in (3.13), we proceed as follows. First, we recall the definition of the Weyl–Heisenberg coherent states

$$|x,p\rangle = \exp\left(-\mathrm{i}\frac{x\cdot P}{\hbar}\right)\exp\left(\mathrm{i}\frac{p\cdot X}{\hbar}\right)|0,0\rangle$$
$$= \exp\left(-\frac{\mathrm{i}}{\hbar}(x\cdot P - p\cdot X)\right)\exp\left(-\mathrm{i}\frac{(x\cdot p)}{2\hbar}\right)|0,0\rangle \tag{4.1a}$$

where $|0,0\rangle$ is the ground state of H in (3.1), i.e.

$$\langle r,\chi|0,0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp -\frac{m\omega}{2\hbar}r^2 \tag{4.1b}$$

(see (3.11a) for $N=0$). We recall that [9]

$$\int |x,p\rangle\langle x,p|\frac{\mathrm{d}x\,\mathrm{d}p}{(2\pi\hbar)^2} = 1. \tag{4.2}$$

Note furthermore that

$$\exp\left(\mathrm{i}\frac{\phi J_3}{\hbar}\right)|x,p\rangle = |x',p'\rangle \tag{4.3a}$$

where

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\phi/2 & \sin\phi/2 \\ -\sin\phi/2 & \cos\phi/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{4.3b}$$

and similarly for (p_1', p_2'). Moreover

$$\exp\left(\mathrm{i}\frac{\theta J_2}{\hbar}\right)|x',p'\rangle = \exp\left(\mathrm{i}\frac{\theta J_2}{\hbar}\right)\exp\left(-\frac{\mathrm{i}}{\hbar}(x'\cdot P - p'\cdot X)\right)$$
$$\times\exp\left(-\mathrm{i}\frac{(x'\cdot p')}{2\hbar}\right)\exp\left(-\mathrm{i}\frac{\theta J_2}{\hbar}\right)|0,0\rangle \tag{4.4a}$$

since

$$\exp\left(-\mathrm{i}\frac{\theta J_2}{\hbar}\right)|0,0\rangle = |0,0\rangle. \tag{4.4b}$$

A simple calculation yields

$$\exp\left(\mathrm{i}\frac{\theta J_2}{\hbar}\right)\begin{pmatrix} X_1 \\ P_2 \end{pmatrix}\exp\left(-\mathrm{i}\frac{\theta J_2}{\hbar}\right) = \begin{pmatrix} \cos\theta/2 & (m\omega)^{-1}\sin\theta/2 \\ -(m\omega)\sin\theta/2 & \cos\theta/2 \end{pmatrix}\begin{pmatrix} X_1 \\ P_2 \end{pmatrix} \tag{4.5a}$$

$$\exp\left(\mathrm{i}\frac{\theta J_2}{\hbar}\right)\begin{pmatrix} X_2 \\ P_1 \end{pmatrix}\exp\left(-\mathrm{i}\frac{\theta J_2}{\hbar}\right) = \begin{pmatrix} \cos\theta/2 & (m\omega)^{-1}\sin\theta/2 \\ -(m\omega)\sin\theta/2 & \cos\theta/2 \end{pmatrix}\begin{pmatrix} X_2 \\ P_1 \end{pmatrix} \tag{4.5b}$$

Using (4.5) in (4.4*a*) gives

$$\exp\left(\mathrm{i}\frac{\theta J_2}{\hbar}\right)\exp\left(\mathrm{i}\frac{\phi J_3}{\hbar}\right)|x,p\rangle = \exp\left(-\frac{\mathrm{i}}{2}\frac{x'\cdot p'}{\hbar}\right)\exp\left(\frac{\mathrm{i}}{2}\frac{x''\cdot p''}{\hbar}\right)|x'',p''\rangle \tag{4.6a}$$

with

$$\begin{pmatrix} p_2'' \\ x_1'' \end{pmatrix} = \begin{pmatrix} \cos\theta/2 & (m\omega)\sin\theta/2 \\ -(m\omega)^{-1}\sin\theta/2 & \cos\theta/2 \end{pmatrix}\begin{pmatrix} p_2' \\ x_1' \end{pmatrix} \tag{4.6b}$$

and similarly for $\binom{p_1''}{x_2''}$. Note that (x'',p'') is obtained by integrating the Hamiltonian flow corresponding to J_2 in (2.3*b*) with initial condition (x',p'). Similarly (x',p') is obtained by integrating the Hamiltonian flow corresponding to J_3 with initial condition (x,p). Now, we can compute

$$\begin{aligned}
\langle x,p|\tfrac{1}{2}N;\theta,\phi\rangle &= \langle x'',p''|\tfrac{1}{2}N,\tfrac{1}{2}N\rangle \exp\left(\frac{\mathrm{i}}{2}\frac{x'\cdot p'}{\hbar}\right)\exp\left(-\frac{\mathrm{i}}{2}\frac{x''\cdot p''}{\hbar}\right) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)\int \mathrm{d}^2 y\, \exp\left(-\frac{m\omega}{2\hbar}y^2\right)\exp(\mathrm{i}N\chi)\frac{1}{\sqrt{N!}}\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}N}|y|^N \\
&\quad\times \exp\left(-\frac{m\omega}{2\hbar}(\boldsymbol{y}-\boldsymbol{x}'')^2\right)\exp\left(-\mathrm{i}\frac{\boldsymbol{p}''\cdot(\boldsymbol{y}-\boldsymbol{x}'')}{\hbar}\right) \\
&\quad\times \exp\left(\frac{\mathrm{i}}{2}\frac{x'\cdot p'}{\hbar}\right)\exp\left(-\frac{\mathrm{i}}{2}\frac{x''\cdot p''}{\hbar}\right) \\
&= \frac{1}{\sqrt{N!}\pi}\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}N+1}\exp\left(\frac{\mathrm{i}}{2}\frac{\boldsymbol{p}''\cdot\boldsymbol{x}''}{\hbar}\right) \\
&\quad\times \exp\left(-\frac{m\omega}{2\hbar}(\boldsymbol{x}'')^2\right)\exp\left(\frac{\mathrm{i}}{2}\frac{\boldsymbol{p}'\cdot\boldsymbol{x}'}{\hbar}\right) \\
&\quad\times \int \mathrm{d}^2\boldsymbol{y}\,|y|^N \exp\left(-\frac{m\omega}{\hbar}y^2\right)\exp\left(\frac{1}{\hbar}[m\omega\boldsymbol{x}''-\mathrm{i}\boldsymbol{p}'']\cdot\boldsymbol{y}\right)\exp(\mathrm{i}N\chi) \\
&= \frac{1}{\pi\sqrt{N!}}\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}N+1}\exp\left(\frac{\mathrm{i}}{2}\frac{\boldsymbol{p}''\cdot\boldsymbol{x}''}{\hbar}\right)\exp\left(-\frac{m\omega}{2\hbar}(\boldsymbol{x}'')^2\right) \\
&\quad\times \exp\left(\frac{\mathrm{i}}{2}\frac{\boldsymbol{p}'\cdot\boldsymbol{x}'}{\hbar}\right)\int \mathrm{d}^2 z\, \exp\left[-\frac{m\omega}{\hbar}|z|^2\right. \\
&\quad\left. + \frac{1}{2\hbar}(w_1-\mathrm{i}w_2)z + \frac{1}{2\hbar}(w_1+\mathrm{i}w_2)z^*\right]z^N
\end{aligned} \tag{4.7a}$$

where

$$z = y_1 + \mathrm{i}y_2 \tag{4.7b}$$

$$\boldsymbol{w} = m\omega\boldsymbol{x}'' - \mathrm{i}\boldsymbol{p}''. \tag{4.7c}$$

Using

$$\int \frac{d^2 z}{\pi} \exp\left[\zeta|z|^2 + \mu z + \nu z^*\right] = \left(-\frac{1}{\zeta}\right) \exp\left(-\frac{\mu\nu}{\zeta}\right) \qquad \mathrm{Re}\,\zeta < 0 \tag{4.7d}$$

we obtain

$$\int \frac{d^2 z}{\pi} z^N \exp\left[\zeta|z|^2 + \mu z + \nu z^*\right] = \left(-\frac{1}{\zeta}\right)\left(-\frac{\nu}{\zeta}\right)^N \exp\left(-\frac{\mu\nu}{\zeta}\right). \tag{4.7e}$$

Hence one computes readily

$$\begin{aligned}\langle x, p|\tfrac{1}{2}N; \theta, \phi\rangle = {} & \sqrt{\frac{1}{N!}}\left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}N}\left(\frac{1}{2\hbar}\right)^N \\ & \times \exp\left[\frac{i}{2}\frac{\boldsymbol{p}'\cdot\boldsymbol{x}'}{\hbar} - \frac{m\omega}{4\hbar}|\boldsymbol{x}''|^2 - \frac{1}{4m\omega\hbar}|\boldsymbol{p}''|^2\right] \\ & \times \left(m\omega x_1'' + p_2'' + i(m\omega x_2'' - p_1'')\right)^N .\end{aligned} \tag{4.8}$$

To study the semiclassical behaviour of this wavefunction, we shall use

$$E = \hbar\omega(N+1) \tag{4.9a}$$

and let $N \to \infty, \hbar \to 0$, keeping E fixed. We have

$$|\langle x, p|\tfrac{1}{2}N; \theta, \phi\rangle|^2 = \frac{1}{N!}\left(\frac{N+1}{2E}\right)^N (H'' + 2\omega J_3'')^N \exp\left(-\frac{H''}{E}\right)(N+1). \tag{4.9b}$$

Note first that it will be much easier to study the expression (4.9) as $N \to \infty$, than the position space wavefunction in (3.13). Indeed, the complicated dependence of the latter on N is self-evident.

To formulate our main result, we introduce for each locally integrable $f : \mathbb{R}^4 \to \mathbb{R}$ a function

$$\langle f\rangle : j \in R^3 \to \frac{1}{T}\int_0^T d\tau\, f(x(\tau), p(\tau)) \tag{4.10}$$

where $T = 2\pi/\omega$, and the integral is taken over the trajectory $(x(\tau), p(\tau))$ determined by $J_i(x(\tau), p(\tau)) = j_i$. Then we have

Theorem 4.1. Let $f : \mathbb{R}^4 \to \mathbb{R}$ in $C^3(\mathbb{R}^4), E > 0, \theta \in [0, \pi]$, and $\phi \in [0, 2\pi)$ be given. Define (see (2.10))

$$j_1 = (E/2\omega)\sin\theta\cos\phi \tag{4.11a}$$

$$j_2 = (E/2\omega)\sin\theta\sin\phi \tag{4.11b}$$

$$j_3 = (E/2\omega)\cos\theta. \tag{4.11c}$$

Suppose there exists an $R > 0$ so that $\int f(x,p)\exp(-RH(x,p))\,\mathrm{d}x\,\mathrm{d}p < \infty$, then

$$\int f(x,p)|\langle x,p|\tfrac{1}{2}N;\theta,\phi\rangle|^2\frac{\mathrm{d}x\,\mathrm{d}p}{(2\pi\hbar)^2} - \langle f\rangle(j) + \frac{\hbar\omega}{E}\left[\frac{1}{2}j\cdot\nabla_j + \frac{1}{2}(j\cdot\nabla_j)^2 - j^2\Delta_j\right]\langle f\rangle(j) = \mathrm{O}(\hbar^2). \qquad (4.12)$$

Remarks. (i) The theorem states in particular that the phase space distribution $|\langle x,p|\frac{1}{2}N;\theta,\phi\rangle|^2$ converges, as $\hbar \to 0$, to a delta function on the classical trajectory.

(ii) Introducing

$$F_\hbar(X,P) \equiv \int \frac{\mathrm{d}x\,\mathrm{d}p}{(2\pi\hbar)^2} f(x,p)|x,p\rangle\langle x,p| \qquad (4.13)$$

equation (4.12) can be written as

$$\langle\tfrac{1}{2}N;\theta,\phi|F_\hbar(X,P)|\tfrac{1}{2}N;\theta,\phi\rangle - \langle f\rangle(j) + \frac{\hbar\omega}{E}\left[\frac{1}{2}j\cdot\nabla_j + \frac{1}{2}(j\cdot\nabla_j)^2 - j^2\Delta_j\right]\langle f\rangle(j) = \mathrm{O}(\hbar^2). \qquad (4.12')$$

Thinking of $F_\hbar$ as the quantization of the classical observable f, equation (4.13) states that the expected value of $F_\hbar$ in the state $|\frac{1}{2}N;\theta,\phi\rangle$ converges, as $\hbar \to 0$, to the time average of f along the classical trajectory. Remark that f is—by definition—nothing but the anti-Wick symbol of $F_\hbar$ [9]. The result of the theorem is therefore easily generalized to operators $F_\hbar$ having an anti-Wick symbol $f_\hbar$ that permits an asymptotic expansion in $\hbar$ with f as its leading term.

(iii) All the higher order terms in $\hbar$ can also be computed using the results of the appendix, but they do not seem to have an obvious or simple form. We remark nevertheless that they only depend on f through $\langle f\rangle$, i.e. on the way the average value of f over one period depends on the trajectory.

(iv) If f is supported away from the classical trajectory, then the proof of the theorem shows that $I(\hbar)$ is exponentially small in $\hbar^{-1}$: see (4.19).

Proof. Introduce first the functions

$$k_1(x,p) = \frac{H(x,p) + 2\omega J_3(x,p)}{2E} \geqslant 0 \qquad (4.14a)$$

$$k_2(x,p) = \frac{H(x,p) - 2\omega J_3(x,p)}{2E} \geqslant 0. \qquad (4.14b)$$

Then we have, from (4.9),

$$(2\pi\hbar)^{-2}|\langle x,p|\tfrac{1}{2}N;\theta;\phi\rangle|^2 = \left(\frac{\omega}{2\pi E}\right)^2 [C_N \exp(N(\log k_1'' - k_1'' + 1))\exp(1 - k_1'')] \times [(N+1)\exp - k_2''(N+1)]. \qquad (4.15a)$$

with

$$C_N = \frac{(N+1)^{(N+1)} e^{-(N+1)}}{N!} \tag{4.15b}$$

and $k_i'' = k_i(x'', p'')$. We now perform the canonical change of variables $(x, p) \to (x'', p'')$ in the integral

$$I(\hbar) = \int \frac{\mathrm{d}x\,\mathrm{d}p}{(2\pi\hbar)^2} f(x,p) |\langle x, p | \tfrac{1}{2}N; \theta; \phi\rangle|^2 \tag{4.16}$$

and introduce $\phi : \mathbb{R}^4 \to \mathbb{R}^4, \phi(x'', p'') = (x, p)$, defined by (4.3*b*) and (4.6*b*). We then obtain (dropping the primes)

$$I(\hbar) = \int \mathrm{d}x\,\mathrm{d}p\,(f \circ \phi)(x,p) \times \left(\frac{\omega}{2\pi E}\right)^2 [C_N \exp(N(\log k_1 - k_1 + 1)) \exp(1 - k_1)] \\ \times [(N+1)\exp(-k_2(N+1))] \,. \tag{4.17}$$

Remarking that the two factors in square brackets in (4.17) are delta-function sequences (see the appendix), the leading order term in (4.12) can immediately be obtained from (4.17) by a formal computation.

We shall control $I(\hbar)$ by cutting the region of integration in two pieces:

$$A = \{(x,p) \in \mathbb{R}^4 | E(1-\epsilon_2) < H(x,p) < E(1+\epsilon_1), J_1^2(x,p) < \delta^2(E/2\omega)^2 \\ \text{for some } \tfrac{1}{2} > \epsilon, \epsilon_2 > 0 \text{ and } \delta^2 \leqslant \tfrac{1}{2}(1-\epsilon_2)^2, J_3 > 0\} \tag{4.18a}$$

$$B = \mathbb{R}^4 \setminus A. \tag{4.18b}$$

Correspondingly we have $I(\hbar) = I_A(\hbar) + I_B(\hbar)$. Region A is close to the anti-clockwise circular trajectory of energy E and B is away from this trajectory. We need to prove that the integrand in (4.17) concentrates in A as $\hbar \to 0$.

We first show that there exist constants $C_f > 0$ and $N_0(R, \epsilon_1, \epsilon_2) > 0$ so that, for $N > N_0(R, \epsilon_1, \epsilon_2)$, we have

$$I_B(\hbar) \leqslant C_f \exp(-(\lambda/2)N) \tag{4.19}$$

where λ is a constant depending on ϵ_1 and on ϵ_2 so that $\lambda(\epsilon_1, \epsilon_2) \to 0$ when ϵ_1 or ϵ_2 tend to zero. Equation (4.19) implies that the phase space density $|\langle x, p|\tfrac{1}{2}N; \theta; \phi\rangle|^2$ is exponentially small in N away from the classical trajectory.

To establish (4.19), we cut region B itself in four parts (with ϵ_1, ϵ_2 and δ as in (4.18*a*)):

$(B1)$ $\quad k_1 + k_2 - 1 > \epsilon_1$

$(B2)$ $\quad k_1 + k_2 - 1 < -\epsilon_2$

$(B3)$ $\quad -\epsilon_2 < k_1 + k_2 - 1 < \epsilon_1 \quad$ and $\quad J_1^2 + J_2^2 > \delta^2(E/2\omega)^2$

$(B4)$ $\quad -\epsilon_2 < k_1 + k_2 - 1 < \epsilon_1, J_1^2 + J_2^2 < \delta^2(E/2\omega)^2 \quad$ and $\quad J_3 < 0$

and write

$$I_B(\hbar) = I_B^1(\hbar) + I_B^2(\hbar) + I_B^3(\hbar) + I_B^4(\hbar). \tag{4.20}$$

We need to estimate

$$k_1^N \exp[-(N+1)(k_1-1)] \exp[-k_2(N+1)] \exp[R(k_1+k_2)]$$

in each region. We start with $(B1)$. Here the factor k_1^N can blow up with N and therefore needs to be controlled by the exponential. We find

$$\begin{aligned} k_1^N \exp[-(N+1)(k_1-1)] &\exp[-k_2(N+1)] \exp[R(k_1+k_2)] \\ &= e^R \exp[N \log k_1 - (N+1-R)k_1 \\ &\quad + (N+1-R) - (N+1-R)k_2] \\ &= e^R \exp\left\{N\left[\log k_1 - \left(1+\frac{1-R}{N}\right)(k_1+k_2-1)\right]\right\} \end{aligned} \tag{4.21}$$

Set $x = k_1 + k_2 - 1$ and remark that on $(B1)$ $x > \epsilon_1, x \geqslant k_1 - 1$, so that

$$\leqslant e^R \exp\left\{N\left[\log(x+1) - \left(1-\frac{R-1}{N}\right)x\right]\right\}. \tag{4.22}$$

It is easy to see that the function in the exponent decreases in x provided $x > (R-1)/[N-(R-1)]$, so that for all N such that

$$\frac{R-1}{N-(R-1)} < \epsilon_1 \qquad \text{i.e. } N > (R-1)\frac{1+\epsilon_1}{\epsilon_1}$$

we have

$$\leqslant e^R \exp N\left[\log(\epsilon_1+1) - \left(1-\frac{R-1}{N}\right)\epsilon_1\right].$$

Setting

$$\lim_{N\to\infty}\left[\log(\epsilon_1+1) - \left(1-\frac{R-1}{N}\right)\epsilon_1\right] = \log(\epsilon_1+1) - \epsilon_1 = -2\alpha \tag{4.23}$$

we conclude

$$k_1^N \exp[-(N+1)(k_1-1)] \exp[-k_2(N+1)] \exp[R(k_1+k_2)] \leqslant e^R e^{-\alpha N} \tag{4.24}$$

for N large enough (depending on ϵ_1).

Using (4.24), we conclude that there exists an $N_0(R,\epsilon_1)$ so that for $N > N_0(R,\epsilon_1)$ one has

$$I_B^1(\hbar) \leqslant \left(\frac{\omega}{2\pi E}\right)^2 (N+1) C_N e^{-\alpha N} C_R \int (f\circ\phi)(x,p) e^{-RH}\, dx\, dp \tag{4.25}$$

which yields the desired estimate provided one uses Stirling's formula to control C_N, i.e.

$$\frac{\sqrt{2\pi(N+1)}(N+1)^{(N+1)}e^{-(N+1)}}{(N+1)!} = O(1).$$

Note that $\alpha \to 0$ as ϵ_1 goes to zero (see (4.23)).

To control $I_B^2(\hbar)$ we note that $k_1 + k_2 \langle 1 - \epsilon_2$ implies $k_1 \langle 1 - \epsilon_2$ and so

$$\begin{aligned}
k_1^N \exp[-(N+1)(k_1-1)]\exp[-k_2(N+1)]\exp[R(k_1+k_2)] \\
&\leqslant k_1^{(N-R)}\exp[-N(k_1-1)]\exp[R(k_1-1)]\exp[-(N+1-R)k_2] \\
&\quad \times e^R \exp(1-k_1) \\
&\leqslant \exp(R+1)\exp[(N-R)(\log k_1 - (k_1-1))]
\end{aligned}$$

provided $N+1 > R$. But $\log k_1 - (k_1 - 1)$ reaches its maximum at $k_1 = 1$, where it is zero. So

$$\leqslant C_R \exp\left[-N\left(1-\frac{R}{N}\right)\delta\right] \tag{4.26}$$

for some $\delta > 0$, depending on ϵ_2. This establishes the estimate for $I_B^2(\hbar)$.

To control $I_B^3(\hbar)$, note that in the region $(B3)$, we have

$$\begin{aligned}
k_1^N \exp[-(N+1)(k_1-1)]\exp[-k_2(N+1)]\exp[R(k_1+k_2)] \\
&\leqslant C_R \exp[-k_2(N+1)].
\end{aligned} \tag{4.27}$$

Moreover, from (4.14) and (2.5),

$$\begin{aligned}
k_2 &= \frac{H}{2E} - \frac{1}{2E}\sqrt{H^2 - \omega^2(J_1^2 + J_2^2)} \\
&\geqslant \frac{H}{2E} - \frac{1}{2E}\sqrt{H^2 - \delta E^2}.
\end{aligned}$$

Hence there exists a $\gamma > 0$, depending on $\delta, \epsilon_1, \epsilon_2$, so that, on $(B3)$, we have

$$k_2 \geqslant \gamma. \tag{4.28}$$

Inserting (4.28) into (4.27), we obtain the desired estimate for $I_B^3(\hbar)$. Finally, similar arguments provide the estimate for $(B4)$; we omit the details. This ends the proof of (4.19).

Thanks to (4.19) we can assume, when estimating $I(\hbar)$, that $f \circ \phi$ is compactly supported inside (A). Indeed, any contribution to $I(\hbar)$ coming from (B) is exponentially small in $\hbar^{-1}$ and does not therefore contribute to (4.12). To control $I_A(\hbar)$, we start by rewriting the volume element by using a convenient system of coordinates on region (A) (see (4.38)). We first perform a change of variables

$$(x, p) \to (H, J_1, J_2, \tau) \in \mathbb{R}^4 \tag{4.29}$$

where $\tau(x,p) \in [0, T = 2\pi/\omega)$ is defined as follows. Each point (x,p) in region (A) lies on exactly one flow line of X_H; τ is the time elapsed from the point on this trajectory where $x_2 = 0, x_1 > 0$. Clearly one has

$$i_{X_H}\,\mathrm{d}\tau = 1. \tag{4.30}$$

It is then clear that (4.29) is a diffeomorphism on (A). We know that there exists a function $K(H, J_1, J_2, \tau)$, so that

$$\tfrac{1}{2}(\omega \wedge \omega) = K(H, J_1, J_2, \tau)\,\mathrm{d}\tau \wedge \mathrm{d}H \wedge \mathrm{d}J_1 \wedge \mathrm{d}J_2. \tag{4.31}$$

Hence

$$i_{X_H}(\tfrac{1}{2})(\omega \wedge \omega) = K(H, J_1, J_2, \tau)\,\mathrm{d}H \wedge \mathrm{d}J_1 \wedge \mathrm{d}J_2. \tag{4.32}$$

so that

$$i_{X_{J_2}} i_{X_{J_1}} i_{X_H}(\tfrac{1}{2})(\omega \wedge \omega) = K(H, J_1, J_2, \tau) J_3^2\,\mathrm{d}H. \tag{4.33}$$

On the other hand

$$i_{X_H}(\tfrac{1}{2})(\omega \wedge \omega) = \tfrac{1}{2}(\mathrm{d}H \wedge \omega + \omega \wedge \mathrm{d}H)$$

so that

$$i_{X_{J_1}} i_{X_H}(\tfrac{1}{2})(\omega \wedge \omega) = \mathrm{d}J_1 \wedge \mathrm{d}H$$

and

$$i_{X_{J_2}} i_{X_{J_1}} i_{X_H}(\tfrac{1}{2})(\omega \wedge \omega) = J_3\,\mathrm{d}H. \tag{4.34}$$

From (4.33)–(4.34), we can conclude

$$K(H, J_1, J_2, \tau) = J_3^{-1}. \tag{4.35}$$

Note that $J_3 \neq 0$ on all of (A), so that (4.35) is well defined. Note also that K does not depend on τ. Introducing polar coordinates (j, Θ) in the J_1, J_2 plane by

$$J_1 = j \cos\Theta \tag{4.36a}$$

$$J_2 = j \sin\Theta \tag{4.36b}$$

we note that (4.14) and (2.5) imply

$$j = \frac{E}{\omega}(k_1 k_2)^{1/2} \tag{4.37a}$$

and

$$H = E(k_1 + k_2) \tag{4.37b}$$

so that we can finally write the volume element as

$$\frac{1}{2}(\omega \wedge \omega) = \frac{E^2}{\omega} \mathrm{d}k_1 \wedge \mathrm{d}k_2 \wedge \mathrm{d}\Theta \wedge \mathrm{d}\tau \tag{4.38}$$

in terms of the independent coordinates (k_1, k_2, Θ, τ) on (A).

To obtain the desired estimate (4.12), we insert (4.38) in (4.17), express the function $(f \circ \phi)$ in the integrand in terms of the coordinates (τ, J_1, J_2, J_3) on (A), and expand in a Taylor series about $J_1 = 0, J_2 = 0, J_3 = E/2\omega$ as follows. First, introduce the notation

$$(f \circ \phi)(x, p) = \widehat{f}(\tau, J_1, J_2, J_3). \tag{4.39}$$

We then get, from (4.17), and using the notation of the appendix,

$$I(\hbar) = \left(\frac{\omega}{4\pi^2}\right) \int_A \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}\Theta\, \mathrm{d}\tau\, \widehat{f}(\tau, J_1, J_2, J_3) A_N(k_1) B_N(k_2) + \mathrm{O}(\mathrm{e}^{-\mu N}) \tag{4.40}$$

for some $\mu > 0$. Now, since $\widehat{f}$ is supported inside region A, we can extend the integral over the full range of the variables $0 \leqslant k_1, k_2 < \infty$. Taylor expansion of $\widehat{f}$ about the point $(\tau, J) = (\tau, 0, 0, E/2\omega)$ gives

$$\begin{aligned}
\widehat{f}(\tau, J) = {} & \widehat{f}(\tau, 0, 0, E/2\omega) + \frac{\partial \widehat{f}}{\partial J_1}\left(\tau, 0, 0, \frac{E}{2\omega}\right) J_1 + \frac{\partial \widehat{f}}{\partial J_2}\left(\tau, 0, 0, \frac{E}{2\omega}\right) J_2 \\
& + \frac{\partial \widehat{f}}{\partial J_3}\left(\tau, 0, 0, \frac{E}{2\omega}\right)\left(J_3 - \frac{E}{2\omega}\right) + \frac{\partial^2 \widehat{f}}{\partial J_1 \partial J_2}\left(\tau, 0, 0, \frac{E}{2\omega}\right) J_1 J_2 \\
& + \frac{\partial^2 \widehat{f}}{\partial J_1 \partial J_3}\left(\tau, 0, 0, \frac{E}{2\omega}\right) J_1 \left(J_3 - \frac{E}{2\omega}\right) \\
& + \frac{\partial^2 \widehat{f}}{\partial J_2 \partial J_3}\left(\tau, 0, 0, \frac{E}{2\omega}\right) J_2 \left(J_3 - \frac{E}{2\omega}\right) + \frac{1}{2}\frac{\partial^2 \widehat{f}}{\partial J_1^2}\left(\tau, 0, 0, \frac{E}{2\omega}\right) J_1^2 \\
& + \frac{1}{2}\frac{\partial^2 \widehat{f}}{\partial J_2^2}\left(\tau, 0, 0, \frac{E}{2\omega}\right) J_2^2 + \frac{1}{2}\frac{\partial^2 \widehat{f}}{\partial J_3^2}\left(\tau, 0, 0, \frac{E}{2\omega}\right)\left(J_3 - \frac{E}{2\omega}\right)^2 \\
& + R_3(\tau, J).
\end{aligned} \tag{4.41}$$

Inserting (4.41) into (4.40) and using (4.36) and (4.37*a*), together with the notation (4.10), and the results of the appendix, one readily computes

$$\begin{aligned}
I(\hbar) = {} & \left[\langle \widehat{f} \rangle - \frac{1}{N+1}\frac{E}{2\omega}\frac{\partial \langle \widehat{f} \rangle}{\partial J_3} + \frac{1}{N+1}\left(\frac{E}{2\omega}\right)^2 \left(\frac{\partial^2 \langle \widehat{f} \rangle}{\partial J_1^2} + \frac{\partial^2 \langle \widehat{f} \rangle}{\partial J_2^2} + \frac{1}{2}\frac{\partial^2 \langle \widehat{f} \rangle}{\partial J_3^2}\right)\right] \\
& \times \left(\tau, 0, 0, \frac{E}{2\omega}\right) + \int \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}\Theta\, \mathrm{d}\tau\, R_3(\tau, J) + \mathrm{O}(\hbar^2).
\end{aligned} \tag{4.42}$$

Using the results of the appendix, one shows that the integral of the rest term is of order $(N+1)^{-2}$. Equation (4.12) follows then from (4.42), if one remarks that

$$\langle \widehat{f} \rangle \left(0, 0, \frac{E}{2\omega}\right) = \langle f \circ \phi \rangle \left(0, 0, \frac{E}{2\omega}\right) = \langle f \rangle (j_1, j_2, j_3)$$

where j is defined in (4.11) and we recall that the canonical transformation $\phi(x'', p'') = (x, p)$ defined in (4.3*b*) and (4.6*b*) maps the circular trajectory at energy E into the one determined by (θ, ϕ) in (4.11).

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Appendix

Let

$$A_N(k_1) = C_N k_1^N \exp[-(N+1)(k_1-1)] \tag{A.1}$$

$$B_N(k_2) = (N+1)\exp[-k_2(N+1)] \tag{A.2}$$

with

$$C_N = \frac{(N+1)^{(N+1)}\exp[-(N+1)]}{N!}. \tag{A.3}$$

Write

$$\langle (k_1-1)^p\rangle = \int_0^\infty (k_1-1)^p A_N \, \mathrm{d}k_1 \tag{A.4}$$

$$\langle k_2^p\rangle = \int_0^\infty k_2^p B_N \, \mathrm{d}k_2 . \tag{A.5}$$

The function A_N is easily seen to reach its maximum at $k_1 = 1$ and, in fact, for $f \in S(\mathbb{R})$,

$$\lim_{N\to\infty}\int_0^\infty f(k_1)A_N(k_1)\,\mathrm{d}k_1 = f(1). \tag{A.6}$$

Similarly,

$$\lim_{N\to\infty}\int_0^\infty f(k_2)B_N(k_2)\,\mathrm{d}k_2 = f(0). \tag{A.7}$$

So A_N and B_N are δ-function sequences centred at 1 and 0 respectively.

To find the asymptotic expansion of the integrals in (A.6) and (A.7), one needs to expand f in a Taylor series about 1 and 0 respectively, and compute

$$\langle k_2^p\rangle = \frac{p!}{(N+1)^p} \tag{A.8}$$

and

$$\langle (k_1-1)^p\rangle = \sum_{l=0}^{p}\binom{p}{l}(-1)^{p-l}\frac{(N+l)!}{N!(N+1)^l}. \tag{A.9}$$

We immediately find

$$\int_0^\infty f(k_2)B_N(k_2)\,\mathrm{d}k_2 = \sum_{p=0}^{n} f^{(p)}(0)(N+1)^{-p} + \mathrm{O}((N+1)^{-(n+1)}). \tag{A.10}$$

The expansion of (A.6) is more difficult because $\langle (k_1-1)^p\rangle$ is not of order $(N+1)^{-p}$, but of lower order, as we now show.

We need to identify, for each positive integer n, the coefficient of $(N+1)^{-n}$ in (A.9). In other words, we want to find a_n^p so that

$$\langle (k_1-1)^p\rangle = \sum_{n=0}^{p} a_n^p (N+1)^{-n}. \tag{A.11}$$

Note first that $\langle (k_1-1)^0\rangle = 1$ and $\langle (k_1-1)\rangle = 0$ are easily computed, so we need only consider $p \geqslant 2$. Moreover, for $p \geqslant 2$, the coefficient of $(N+1)^0$ is zero. This follows easily upon remarking that

$$\frac{(N+l)!}{N!(N+1)^l} = (N+1)^{-l}(N+1)(N+1+1)\ldots(N+1+l-1) \tag{A.12}$$

and noticing that the term in $(N+1)^0$ in (A.12) equals one for all l. So we only need to consider $n \geqslant 1$. For that purpose, we introduce the Stirling numbers of the first kind S_n^m, defined by [6]

$$x(x+1)\ldots(x+(n-1)) = \sum_{m=0}^{n} (-1)^{n-m} S_n^m x^m. \tag{A.13a}$$

Remark that $(n \geqslant 1, 0 \leqslant m < n)$

$$S_n^m \equiv (-1)^{n-m} \sum_{i_j=0}^{n-1} i_1 \ldots i_{n-m} \qquad S_n^n = 1 \tag{A.13b}$$

where the sum extends over every combination of order $(n-m)$ without repetition (i.e. $i_i \neq i_j$ if $i \neq j$) and without permutation. Note that $S_n^0 = 0$. This allows us to write (A.12) as

$$\begin{aligned} \frac{(N+l)!}{N!(N+1)^l} &= \sum_{m=1}^{l} (-1)^{l-m} S_l^m (N+1)^{m-l} \\ &= \sum_{n=0}^{l-1} (-1)^n S_l^{l-n} (N+1)^{-n}. \end{aligned} \tag{A.14}$$

Inserting (A.14) into (A.9), interchanging the order of summation, and remembering that the term in $(N+1)^0$ is zero, we obtain

$$\langle (k_1-1)^p\rangle = \sum_{n=1}^{p-1} \sum_{l=n+1}^{p} \binom{p}{l} (-1)^{p-l}(-1)^n S_l^{l-n}(N+1)^{-n}. \tag{A.15}$$

The Stirling numbers S_l^{l-n} are polynomials in l of order $2n$ (see [6] pp 150, 184, 224). Moreover, they can be shown to vanish if $l < 1 + n$ (note that in that case the sum in (A.13*b*) is empty). Hence we can extend the sum over l in (A.15) down to $l = 1$. We introduce

$$S_l^{l-n} = \sum_{t=1}^{2n} b_t^n l^t$$

(the sum starts at 1 since zero is a root of the polynomial S_l^{l-n}). The coefficient of $(-1)^n(N+1)^{-n}$ in (A.15) can therefore be written as $(n \geqslant 1)$

$$\sum_{t=1}^{2n} b_t^n \sum_{l=1}^{p} \binom{p}{l} (-1)^{p-l} l^t .$$

But from [5]

$$\sum_{l=1}^{p} \binom{p}{l} (-1)^{p-l} l^t = 0 \qquad \text{if } 1 \leqslant t \leqslant p-1.$$

Hence, if $2n \leqslant p-1$, the coefficient of $(N+1)^{-n}$ vanishes. We can therefore rewrite (A.15) as follows

$$\langle (k_1-1)^p \rangle = \sum_{n>(p-1)/2}^{p-1} (-1)^n \left(\sum_{l=n+1}^{p} \binom{p}{l} (-1)^{p-l} S_l^{l-n} \right) (N+1)^{-n}. \qquad \text{(A.16)}$$

(Here the sum starts at the first integer bigger than $(p-1)/2$). To conclude, we rewrite, introducing $i = 2n - p$,

$$\sum_{l=n+1}^{p} \binom{p}{l} (-1)^{p-l} S_l^{l-n} = \sum_{l=n+1}^{2n-i} \binom{2n-i}{l} (-1)^{i+l} S_l^{l-n}$$

$$\stackrel{\text{def}}{\equiv} C_{n,2n-p}.$$

The coefficients $C_{n,2n-p}$ are defined and studied in [6] (see pp 152 and 184), where it is shown that they can be determined recursively. Finally, (A.16) becomes $(p > 1)$

$$\langle (k_1-1)^p \rangle = \sum_{n>(p-1)/2}^{p-1} (-1)^n C_{n,2n-p} (N+1)^{-n}. \qquad \text{(A.17)}$$

Note that $\langle (k_1-1)^p \rangle$ is indeed only of order at most $(N+1)^{-(p-1)/2}$. Now,

$$\begin{aligned} \int_0^\infty & f(k_1) A_N(k_1)\, \mathrm{d}k_1 \\ &= \sum_{p=0}^{2m} \frac{f^{(p)}(1)}{p!} \langle (k_1-1)^p \rangle + R_{2m+1} \\ &= f(1) + \sum_{n=1}^{m-1} (-1)^n \left(\sum_{p=n+1}^{2n} \frac{f^{(p)}(1)}{p!} C_{n,2n-p} \right) \\ &\quad \times (N+1)^{-n} + \mathrm{O}((N+1)^{-m}) \end{aligned} \qquad \text{(A.18)}$$

where we have used the easily established fact that R_{2m+1} is of order $(N+1)^{-m}$. The asymptotic expansion in (A.18) shows that, to get all terms up to order m, one typically has to consider derivatives up to order $2m$, in contrast to the expansion in (A.10), which is considerably simpler. The expansions in (A.10) and (A.17) can be used to get the complete asymptotic expansion of the integral in (4.12). We list some values of $\langle (k_1-1)^p \rangle$ for small p:

$$\langle (k_1-1)^0 \rangle = 1 \qquad \langle (k_1-1)^3 \rangle = \frac{2}{(N+1)^2}$$

$$\langle (k_1-1)^1 \rangle = 0 \qquad \langle (k_1-1)^4 \rangle = \frac{3}{(N+1)^2} + \frac{6}{(N+1)^3}$$

$$\langle (k_1-1)^2 \rangle = \frac{1}{(N+1)} \qquad \langle (k_1-1)^5 \rangle = \frac{20}{(N+1)^3} + \frac{24}{(N+1)^4}$$

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